MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

8.02

Review A: Vector Analysis

Vector Analysis

A.1 Vectors

A.1.1 Introduction

Certain physical quantities such as mass or the absolute temperature at some point only have magnitude. These quantities can be represented by numbers alone, with the appropriate units, and they are called scalars. There are, however, other physical quantities which have both magnitude and direction; the magnitude can stretch or shrink, and the direction can reverse. These quantities can be added in such a way that takes into account both direction and magnitude. Force is an example of a quantity that acts in a certain direction with some magnitude that we measure in newtons. When two forces act on an object, the sum of the forces depends on both the direction and magnitude of the two forces. Position, displacement, velocity, acceleration, force, momentum and torque are all physical quantities that can be represented mathematically by vectors. We shall begin by defining precisely what we mean by a vector.

A.1.2 Properties of a Vector

A vector is a quantity that has both direction and magnitude. Let a vector be denoted by the symbol $\vec{\mathbf{A}}$. The magnitude of $\vec{\mathbf{A}}$ is $|\vec{\mathbf{A}}| \equiv A$. We can represent vectors as geometric objects using arrows. The length of the arrow corresponds to the magnitude of the vector. The arrow points in the direction of the vector (Figure A.1.1).

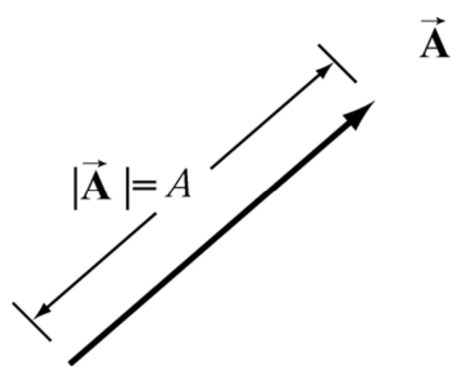

Figure A.1.1 Vectors as arrows.

There are two defining operations for vectors:

(1) Vector Addition: Vectors can be added.

Let $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ be two vectors. We define a new vector, $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$, the "vector addition" of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$, by a geometric construction. Draw the arrow that represents $\vec{\mathbf{A}}$. Place the tail of the arrow that represents $\vec{\mathbf{B}}$ at the tip of the arrow for $\vec{\mathbf{A}}$ as shown in Figure A.1.2(a). The arrow that starts at the tail of $\vec{\mathbf{A}}$ and goes to the tip of $\vec{\mathbf{B}}$ is defined to be the "vector addition" $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$. There is an equivalent construction for the law of vector addition. The vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ can be drawn with their tails at the same point. The two vectors form the sides of a parallelogram. The diagonal of the parallelogram corresponds to the vector $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$, as shown in Figure A.1.2(b).

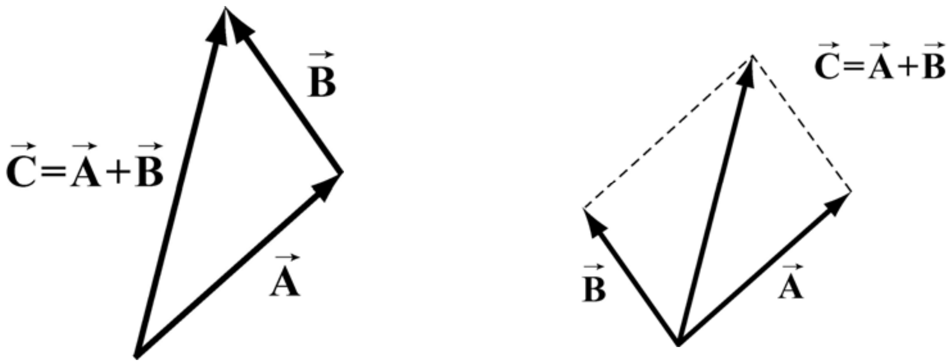

Figure A.1.2 Geometric sum of vectors.

Vector addition satisfies the following four properties:

(i) Commutivity: The order of adding vectors does not matter.

$$\vec{\mathbf{A}} + \vec{\mathbf{B}} = \vec{\mathbf{B}} + \vec{\mathbf{A}} \tag{A.1.1}$$

Our geometric definition for vector addition satisfies the commutivity property (i) since in the parallelogram representation for the addition of vectors, it doesn't matter which side you start with as seen in Figure A.1.3.

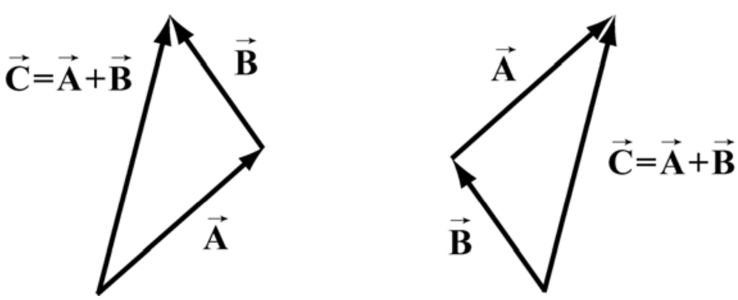

Figure A.1.3 Commutative property of vector addition

(ii) Associativity: When adding three vectors, it doesn't matter which two you start with

$$(\vec{\mathbf{A}} + \vec{\mathbf{B}}) + \vec{\mathbf{C}} = \vec{\mathbf{A}} + (\vec{\mathbf{B}} + \vec{\mathbf{C}}) \tag{A.1.2}$$

In Figure A.1.4(a), we add $(\vec{\mathbf{A}} + \vec{\mathbf{B}}) + \vec{\mathbf{C}}$, while in Figure A.1.4(b) we add $\vec{\mathbf{A}} + (\vec{\mathbf{B}} + \vec{\mathbf{C}})$. We arrive at the same vector sum in either case.

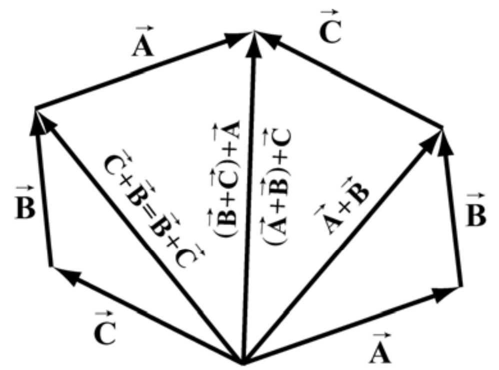

Figure A.1.4 Associative law.

(iii) Identity Element for Vector Addition: There is a unique vector, $\vec{\mathbf{0}}$, that acts as an identity element for vector addition.

This means that for all vectors $\vec{\mathbf{A}}$,

$$\vec{\mathbf{A}}+\vec{\mathbf{0}}=\vec{\mathbf{0}}+\vec{\mathbf{A}}=\vec{\mathbf{A}} \tag{A.1.3}$$

(iv) Inverse element for Vector Addition: For every vector $\vec{\mathbf{A}}$, there is a unique inverse vector

$$(-1)\vec{\mathbf{A}} \equiv -\vec{\mathbf{A}} \tag{A.1.4}$$

such that

$$\vec{\mathbf{A}}+\left(-\vec{\mathbf{A}}\right)=\vec{\mathbf{0}}$$

This means that the vector $-\vec{\mathbf{A}}$ has the same magnitude as $\vec{\mathbf{A}}$, $|\vec{\mathbf{A}}|=|-\vec{\mathbf{A}}|=A$, but they point in opposite directions (Figure A.1.5).

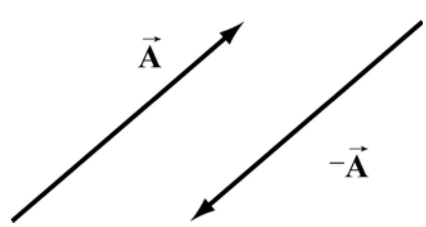

Figure A.1.5 additive inverse.

(2) Scalar Multiplication of Vectors: Vectors can be multiplied by real numbers.

Let $\vec{\mathbf{A}}$ be a vector. Let c be a real positive number. Then the multiplication of $\vec{\mathbf{A}}$ by c is a new vector which we denote by the symbol $c\vec{\mathbf{A}}$. The magnitude of $c\vec{\mathbf{A}}$ is c times the magnitude of $\vec{\mathbf{A}}$ (Figure A.1.6a),

$$cA = Ac \tag{A.1.5}$$

Since $c > 0$, the direction of $c\vec{\mathbf{A}}$ is the same as the direction of $\vec{\mathbf{A}}$. However, the direction of $-c\vec{\mathbf{A}}$ is opposite of $\vec{\mathbf{A}}$ (Figure A.1.6b).

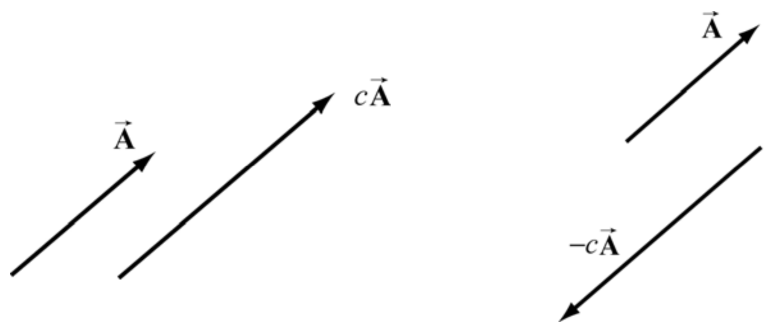

Figure A.1.6 Multiplication of vector $\vec{\mathbf{A}}$ by (a) $c > 0$, and (b) $-c < 0$.

Scalar multiplication of vectors satisfies the following properties:

(i) Associative Law for Scalar Multiplication: The order of multiplying numbers is doesn't matter.

Let b and c be real numbers. Then

$$b(c\vec{\mathbf{A}}) = (bc)\vec{\mathbf{A}} = (cb\vec{\mathbf{A}}) = c(b\vec{\mathbf{A}}) \tag{A.1.6}$$

(ii) Distributive Law for Vector Addition: Vector addition satisfies a distributive law for multiplication by a number.

Let c be a real number. Then

$$c(\vec{\mathbf{A}} + \vec{\mathbf{B}}) = c\vec{\mathbf{A}} + c\vec{\mathbf{B}} \tag{A.1.7}$$

Figure A.1.7 illustrates this property.

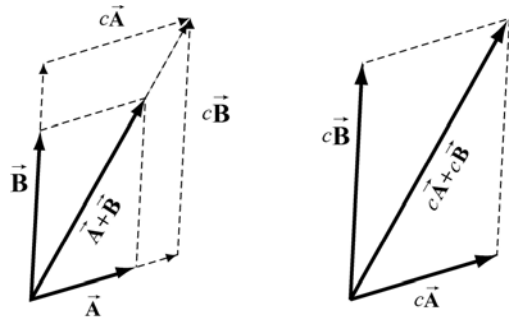

Figure A.1.7 Distributive Law for vector addition.

(iii) Distributive Law for Scalar Addition: The multiplication operation also satisfies a distributive law for the addition of numbers.

Let *b* and *c* be real numbers. Then

$$(b+c)\vec{\mathbf{A}} = b\vec{\mathbf{A}} + c\vec{\mathbf{A}} \tag{A.1.8}$$

Our geometric definition of vector addition satisfies this condition as seen in Figure A.1.8.

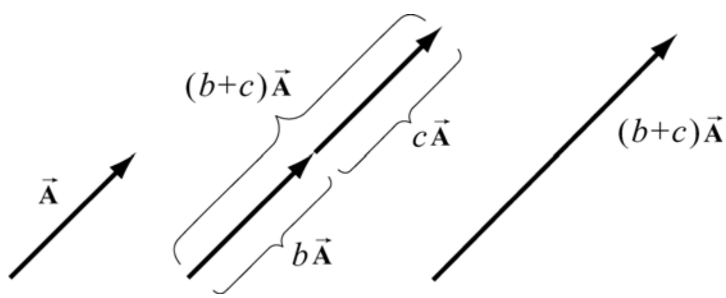

Figure A.1.8 Distributive law for scalar multiplication

(iv) Identity Element for Scalar Multiplication: The number 1 acts as an identity element for multiplication,

$$1\vec{\mathbf{A}} = \vec{\mathbf{A}} \tag{A.1.9}$$

A.1.3 Application of Vectors

When we apply vectors to physical quantities it's nice to keep in the back of our minds all these formal properties. However from the physicist's point of view, we are interested in representing physical quantities such as displacement, velocity, acceleration, force, impulse, momentum, torque, and angular momentum as vectors. We can't add force to velocity or subtract momentum from torque. We must always understand the physical context for the vector quantity. Thus, instead of approaching vectors as formal mathematical objects we shall instead consider the following essential properties that enable us to represent physical quantities as vectors.

(1) Vectors can exist at any point P in space.

(2) Vectors have direction and magnitude.

(3) Vector Equality: Any two vectors that have the same direction and magnitude are equal no matter where in space they are located.

(4) Vector Decomposition: Choose a coordinate system with an origin and axes. We can decompose a vector into component vectors along each coordinate axis. In Figure A.1.9 we choose Cartesian coordinates for the x-y plane (we ignore the z-direction for simplicity but we can extend our results when we need to). A vector $\vec{\mathbf{A}}$ at P can be decomposed into the vector sum,

$$\vec{\mathbf{A}} = \vec{\mathbf{A}}_x + \vec{\mathbf{A}}_y \tag{A.1.10}$$

where $\vec{\mathbf{A}}_x$ is the x-component vector pointing in the positive or negative x-direction, and $\vec{\mathbf{A}}_y$ is the y-component vector pointing in the positive or negative y-direction (Figure A.1.9).

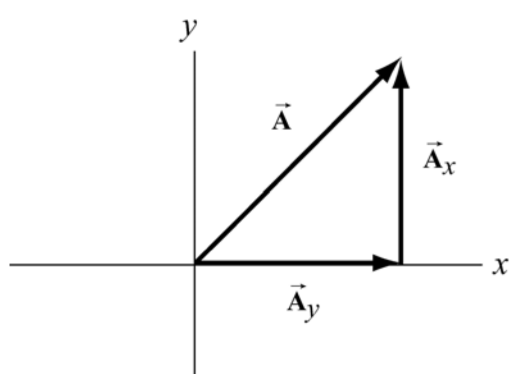

Figure A.1.9 Vector decomposition

(5) Unit vectors: The idea of multiplication by real numbers allows us to define a set of unit vectors at each point in space. We associate to each point P in space, a set of three unit vectors $(\hat{\mathbf{i}},\hat{\mathbf{j}},\hat{\mathbf{k}})$. A unit vector means that the magnitude is one: $|\hat{\mathbf{i}}|=1$, $|\hat{\mathbf{j}}|=1$, and $|\hat{\mathbf{k}}|=1$. We assign the direction of $\hat{\mathbf{i}}$ to point in the direction of the increasing x-coordinate at the point P. We call $\hat{\mathbf{i}}$ the unit vector at P pointing in the $+x$-direction. Unit vectors $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ can be defined in a similar manner (Figure A.1.10).

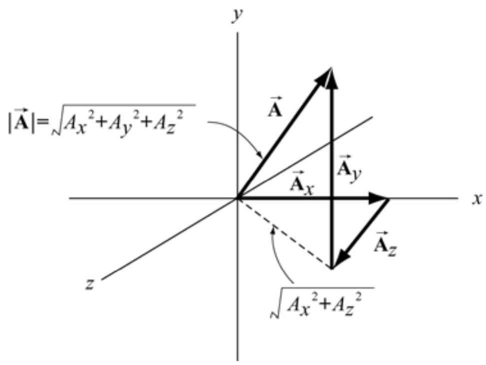

Figure A.1.10 Choice of unit vectors in Cartesian coordinates.

(6) Vector Components: Once we have defined unit vectors, we can then define the x-component and y-component of a vector. Recall our vector decomposition, $\vec{\mathbf{A}} = \vec{\mathbf{A}}_x + \vec{\mathbf{A}}_y$. We can write the *x*-component vector, $\vec{\mathbf{A}}_x$, as

$$\vec{\mathbf{A}}_x = A_x \hat{\mathbf{i}} \qquad \text{(A.1.11)}$$

In this expression the term A_x, (without the arrow above) is called the *x*-component of the vector $\vec{\mathbf{A}}$. The x-component A_x can be positive, zero, or negative. It is not the magnitude of $\vec{\mathbf{A}}_x$ which is given by $(A_x^{\,2})^{1/2}$. Note the difference between the x-component, A_x, and the x-component vector, $\vec{\mathbf{A}}_x$.

In a similar fashion we define the y-component, A_y, and the z-component, A_z, of the vector $\vec{\mathbf{A}}$

$$\vec{\mathbf{A}}_y = A_y \hat{\mathbf{j}}, \quad \vec{\mathbf{A}}_z = A_z \hat{\mathbf{k}} \qquad \text{(A.1.12)}$$

A vector $\vec{\mathbf{A}}$ can be represented by its three components $\vec{\mathbf{A}} = (A_x, A_y, A_z)$. We can also write the vector as

$$\vec{\mathbf{A}} = A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}} \qquad \text{(A.1.13)}$$

(7) Magnitude: In Figure A.1.10, we also show the vector components $\vec{\mathbf{A}} = (A_x, A_y, A_z)$. Using the Pythagorean theorem, the magnitude of the $\vec{\mathbf{A}}$ is,

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} \qquad \text{(A.1.14)}$$

(8) Direction: Let's consider a vector $\vec{\mathbf{A}} = (A_x, A_y, 0)$. Since the z-component is zero, the vector $\vec{\mathbf{A}}$ lies in the *x-y* plane. Let θ denote the angle that the vector $\vec{\mathbf{A}}$ makes in the

counterclockwise direction with the positive x-axis (Figure A.1.12). Then the x-component and y-components are

$$A_x = A\cos\theta, \quad A_y = A\sin\theta \tag{A.1.15}$$

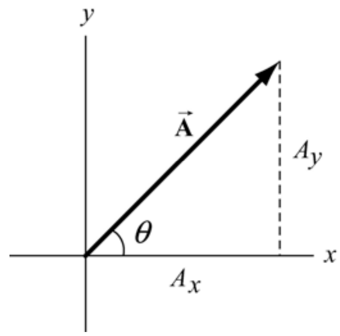

Figure A.1.12 Components of a vector in the *x*-*y* plane.

We can now write a vector in the x-y plane as

$$\vec{\mathbf{A}} = A\cos\theta\,\hat{\mathbf{i}} + A\sin\theta\,\hat{\mathbf{j}} \tag{A.1.16}$$

Once the components of a vector are known, the tangent of the angle θ can be determined by

$$\frac{A_y}{A_x} = \frac{A\sin\theta}{A\cos\theta} = \tan\theta \tag{A.1.17}$$

which yields

$$\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right) \tag{A.1.18}$$

Clearly, the direction of the vector depends on the sign of A_x and A_y. For example, if both $A_x > 0$ and $A_y > 0$, then $0 < \theta < \pi/2$, and the vector lies in the first quadrant. If, however, $A_x > 0$ and $A_y < 0$, then $-\pi/2 < \theta < 0$, and the vector lies in the fourth quadrant.

(9) Vector Addition: Let $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ be two vectors in the *x*-*y* plane. Let θ_A and θ_B denote the angles that the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ make (in the counterclockwise direction) with the positive *x*-axis. Then

$$\vec{\mathbf{A}} = A\cos\theta_A\,\hat{\mathbf{i}} + A\sin\theta_A\,\hat{\mathbf{j}} \qquad \text{(A.1.19)}$$

$$\vec{\mathbf{B}} = B\cos\theta_B\,\hat{\mathbf{i}} + B\sin\theta_B\,\hat{\mathbf{j}} \qquad \text{(A.1.20)}$$

In Figure A.1.13, the vector addition $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$ is shown. Let θ_C denote the angle that the vector $\vec{\mathbf{C}}$ makes with the positive *x*-axis.

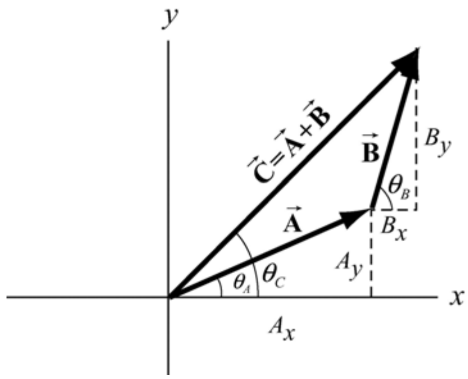

Figure A.1.13 Vector addition with components

Then the components of $\vec{\mathbf{C}}$ are

$$C_x = A_x + B_x, \qquad C_y = A_y + B_y \qquad \text{(A.1.21)}$$

In terms of magnitudes and angles, we have

$$\begin{aligned} C_x &= C\cos\theta_C = A\cos\theta_A + B\cos\theta_B \\ C_y &= C\sin\theta_C = A\sin\theta_A + B\sin\theta_B \end{aligned} \qquad \text{(A.1.22)}$$

We can write the vector $\vec{\mathbf{C}}$ as

$$\vec{\mathbf{C}} = (A_x + B_x)\hat{\mathbf{i}} + (A_y + B_y)\hat{\mathbf{j}} = C(\cos\theta_C\hat{\mathbf{i}} + \sin\theta_C)\hat{\mathbf{j}} \qquad \text{(A.1.23)}$$

A.2 Dot Product

A.2.1 Introduction

We shall now introduce a new vector operation, called the “dot product” or “scalar product” that takes any two vectors and generates a scalar quantity (a number). We shall see that the physical concept of work can be mathematically described by the dot product between the force and the displacement vectors.

Let $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ be two vectors. Since any two non-collinear vectors form a plane, we define the angle θ to be the angle between the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ as shown in Figure A.2.1. Note that θ can vary from 0 to π.

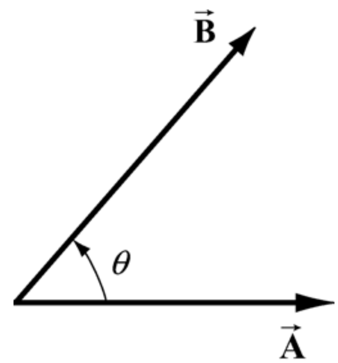

Figure A.2.1 Dot product geometry.

A.2.2 Definition

The dot product $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}$ of the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is defined to be product of the magnitude of the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ with the cosine of the angle θ between the two vectors:

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = AB\cos\theta \tag{A.2.1}$$

Where $A = |\vec{\mathbf{A}}|$ and $B = |\vec{\mathbf{B}}|$ represent the magnitude of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ respectively. The dot product can be positive, zero, or negative, depending on the value of $\cos\theta$. The dot product is always a scalar quantity.

We can give a geometric interpretation to the dot product by writing the definition as

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = (A\cos\theta)B \tag{A.2.2}$$

In this formulation, the term $A\cos\theta$ is the projection of the vector $\vec{\mathbf{A}}$ in the direction of the vector $\vec{\mathbf{B}}$. This projection is shown in Figure A.2.2a. So the dot product is the product of the projection of the length of $\vec{\mathbf{A}}$ in the direction of $\vec{\mathbf{B}}$ with the length of $\vec{\mathbf{B}}$. Note that we could also write the dot product as

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A(B\cos\theta) \tag{A.2.3}$$

Now the term $B\cos\theta$ is the projection of the vector $\vec{\mathbf{B}}$ in the direction of the vector $\vec{\mathbf{A}}$ as shown in Figure A.2.2b.From this perspective, the dot product is the product of the projection of the length of $\vec{\mathbf{B}}$ in the direction of $\vec{\mathbf{A}}$ with the length of $\vec{\mathbf{A}}$.

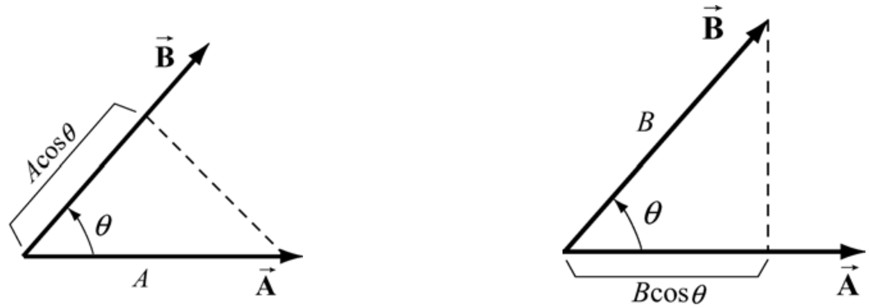

Figure A.2.2a and **A.2.2b** Projection of vectors and the dot product.

From our definition of the dot product we see that the dot product of two vectors that are perpendicular to each other is zero since the angle between the vectors is $\pi/2$ and $\cos(\pi/2) = 0$.

A.2.3 Properties of Dot Product

The first property involves the dot product between a vector $c\vec{\mathbf{A}}$ where c is a scalar and a vector $\vec{\mathbf{B}}$,

(1a) $$c\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = c(\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}) \qquad \text{(A.2.4)}$$

The second involves the dot product between the sum of two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ with a vector $\vec{\mathbf{C}}$,

(2a) $$(\vec{\mathbf{A}} + \vec{\mathbf{B}}) \cdot \vec{\mathbf{C}} = \vec{\mathbf{A}} \cdot \vec{\mathbf{C}} + \vec{\mathbf{B}} \cdot \vec{\mathbf{C}} \qquad \text{(A.2.5)}$$

Since the dot product is a commutative operation

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = \vec{\mathbf{B}} \cdot \vec{\mathbf{A}} \qquad \text{(A.2.6)}$$

the similar definitions hold

(1b) $$\vec{\mathbf{A}} \cdot c\vec{\mathbf{B}} = c(\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}) \qquad \text{(A.2.7)}$$

(2b) $$\vec{\mathbf{C}} \cdot (\vec{\mathbf{A}} + \vec{\mathbf{B}}) = \vec{\mathbf{C}} \cdot \vec{\mathbf{A}} + \vec{\mathbf{C}} \cdot \vec{\mathbf{B}} \qquad \text{(A.2.8)}$$

A.2.4 Vector Decomposition and the Dot Product

With these properties in mind we can now develop an algebraic expression for the dot product in terms of components. Let's choose a Cartesian coordinate system with the vector $\vec{\mathbf{B}}$ pointing along the positive x-axis with positive x-component B_x, i.e., $\vec{\mathbf{B}} = B_x \hat{\mathbf{i}}$. The vector $\vec{\mathbf{A}}$ can be written as

$$\vec{\mathbf{A}} = A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}} \tag{A.2.9}$$

We first calculate that the dot product of the unit vector $\hat{\mathbf{i}}$ with itself is unity:

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = |\hat{\mathbf{i}}||\hat{\mathbf{i}}| \cos(0) = 1 \tag{A.2.10}$$

since the unit vector has magnitude $|\hat{\mathbf{i}}| = 1$ and $\cos(0) = 1$. We note that the same rule applies for the unit vectors in the y and z directions:

$$\hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1 \tag{A.2.11}$$

The dot product of the unit vector $\hat{\mathbf{i}}$ with the unit vector $\hat{\mathbf{j}}$ is zero because the two unit vectors are perpendicular to each other:

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = |\hat{\mathbf{i}}||\hat{\mathbf{j}}| \cos(\pi/2) = 0 \tag{A.2.12}$$

Similarly, the dot product of the unit vector $\hat{\mathbf{i}}$ with the unit vector $\hat{\mathbf{k}}$, and the unit vector $\hat{\mathbf{j}}$ with the unit vector $\hat{\mathbf{k}}$ are also zero:

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = 0 \tag{A.2.13}$$

The dot product of the two vectors now becomes

$$\begin{aligned}
\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} &= (A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}}) \cdot B_x \hat{\mathbf{i}} \\
&= A_x \hat{\mathbf{i}} \cdot B_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} \cdot B_x \hat{\mathbf{i}} + A_z \hat{\mathbf{k}} \cdot B_x \hat{\mathbf{i}} && \text{property (2a)} \\
&= A_x B_x (\hat{\mathbf{i}} \cdot \hat{\mathbf{i}}) + A_y B_x (\hat{\mathbf{j}} \cdot \hat{\mathbf{i}}) + A_z B_x (\hat{\mathbf{k}} \cdot \hat{\mathbf{i}}) && \text{property (1a) and (1b)} \\
&= A_x B_x
\end{aligned} \tag{A.2.14}$$

This third step is the crucial one because it shows that it is only the unit vectors that undergo the dot product operation.

Since we assumed that the vector $\vec{\mathbf{B}}$ points along the positive x-axis with positive x-component B_x, our answer can be zero, positive, or negative depending on the x-component of the vector $\vec{\mathbf{A}}$. In Figure A.2.3, we show the three different cases.

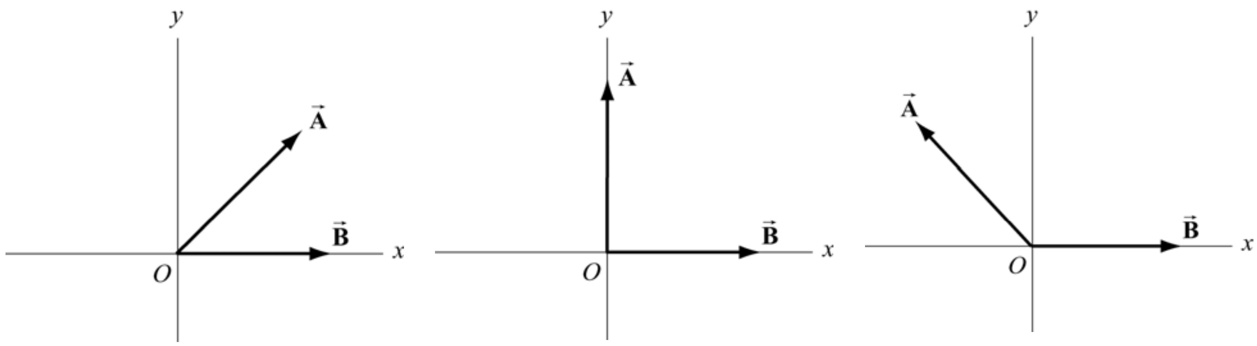

Figure A.2.3 Dot product that is (a) positive, (b) zero or (c) negative.

The result for the dot product can be generalized easily for arbitrary vectors

$$\vec{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}} \tag{A.2.15}$$

and

$$\vec{\mathbf{B}} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}} \tag{A.2.16}$$

to yield

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_x B_x + A_y B_y + A_z B_z \tag{A.2.17}$$

A.3 Cross Product

We shall now introduce our second vector operation, called the "cross product" that takes any two vectors and generates a new vector. The cross product is a type of "multiplication" law that turns our vector space (law for addition of vectors) into a vector algebra (laws for addition and multiplication of vectors). The first application of the cross product will be the physical concept of torque about a point P which can be described mathematically by the cross product of a vector from P to where the force acts, and the force vector.

A.3.1 Definition: Cross Product

Let $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ be two vectors. Since any two vectors form a plane, we define the angle θ to be the angle between the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ as shown in Figure A.3.2.1. The magnitude of the cross product $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ of the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is defined to be product of the magnitude of the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ with the sine of the angle θ between the two vectors,

$$\left|\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right| = AB\sin\theta \tag{A.3.1}$$

where A and B denote the magnitudes of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$, respectively. The angle θ between the vectors is limited to the values $0 \le \theta \le \pi$ insuring that $\sin\theta \ge 0$.

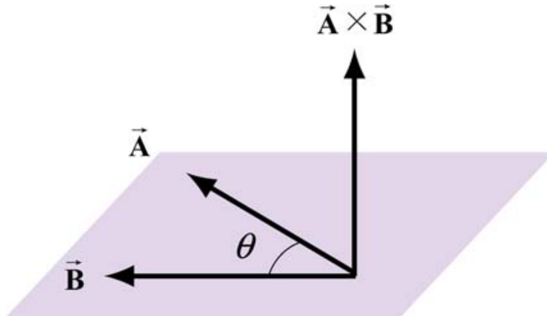

Figure A.3.1 Cross product geometry.

The direction of the cross product is defined as follows. The vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ form a plane. Consider the direction perpendicular to this plane. There are two possibilities, as shown in Figure A.3.1. We shall choose one of these two for the direction of the cross product $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ using a convention that is commonly called the "*right-hand rule*".

A.3.2 Right-hand Rule for the Direction of Cross Product

The first step is to redraw the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ so that their tails are touching. Then draw an arc starting from the vector $\vec{\mathbf{A}}$ and finishing on the vector $\vec{\mathbf{B}}$. Curl your right fingers the same way as the arc. Your right thumb points in the direction of the cross product $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ (Figure A.3.2).

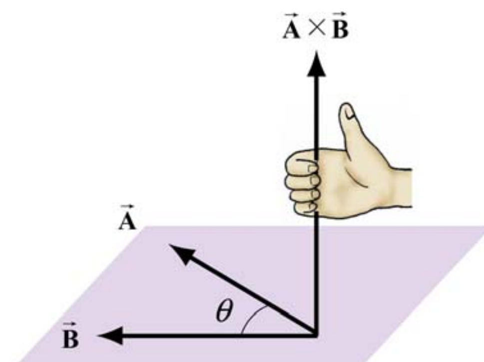

Figure A.3.2 Right-Hand Rule.

You should remember that the direction of the cross product $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ is perpendicular to the plane formed by $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$.

We can give a geometric interpretation to the magnitude of the cross product by writing the definition as

$$\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right| = A\left(B\sin\theta\right) \qquad \text{(A.3.2)}$$

The vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ form a parallelogram. The area of the parallelogram equals the height times the base, which is the magnitude of the cross product. In Figure A.3.3, two different representations of the height and base of a parallelogram are illustrated. As depicted in Figure A.3.3(a), the term $B\sin\theta$ is the projection of the vector $\vec{\mathbf{B}}$ in the direction perpendicular to the vector $\vec{\mathbf{A}}$. We could also write the magnitude of the cross product as

$$\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right| = (A\sin\theta)B \qquad \text{(A.3.3)}$$

Now the term $A\sin\theta$ is the projection of the vector $\vec{\mathbf{A}}$ in the direction perpendicular to the vector $\vec{\mathbf{B}}$ as shown in Figure A.3.3(b).

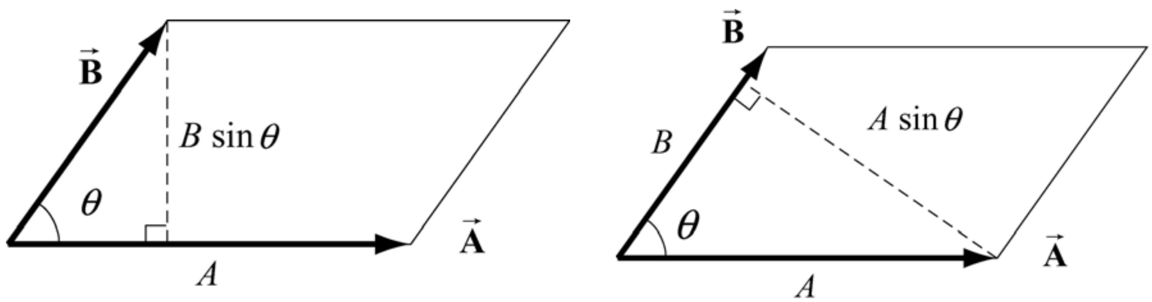

Figure A.3.3 Projection of vectors and the cross product

The cross product of two vectors that are parallel (or anti-parallel) to each other is zero since the angle between the vectors is 0 (or π) and $\sin(0) = 0$ (or $\sin(\pi) = 0$). Geometrically, two parallel vectors do not have any component perpendicular to their common direction.

A.3.3 Properties of the Cross Product

(1) The cross product is anti-commutative since changing the order of the vectors cross product changes the direction of the cross product vector by the right hand rule:

$$\vec{\mathbf{A}}\times\vec{\mathbf{B}} = -\vec{\mathbf{B}}\times\vec{\mathbf{A}} \qquad \text{(A.3.4)}$$

(2) The cross product between a vector $c\vec{\mathbf{A}}$ where c is a scalar and a vector $\vec{\mathbf{B}}$ is

$$c\vec{\mathbf{A}}\times\vec{\mathbf{B}} = c(\vec{\mathbf{A}}\times\vec{\mathbf{B}}) \qquad \text{(A.3.5)}$$

Similarly,

$$\vec{\mathbf{A}}\times c\vec{\mathbf{B}} = c(\vec{\mathbf{A}}\times\vec{\mathbf{B}}) \qquad \text{(A.3.6)}$$

(3) The cross product between the sum of two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ with a vector $\vec{\mathbf{C}}$ is

$$(\vec{\mathbf{A}}+\vec{\mathbf{B}})\times\vec{\mathbf{C}}=\vec{\mathbf{A}}\times\vec{\mathbf{C}}+\vec{\mathbf{B}}\times\vec{\mathbf{C}} \tag{A.3.7}$$

Similarly,

$$\vec{\mathbf{A}}\times(\vec{\mathbf{B}}+\vec{\mathbf{C}})=\vec{\mathbf{A}}\times\vec{\mathbf{B}}+\vec{\mathbf{A}}\times\vec{\mathbf{C}} \tag{A.3.8}$$

A.3.4 Vector Decomposition and the Cross Product

We first calculate that the magnitude of cross product of the unit vector $\hat{\mathbf{i}}$ with $\hat{\mathbf{j}}$:

$$|\hat{\mathbf{i}}\times\hat{\mathbf{j}}|=|\hat{\mathbf{i}}||\hat{\mathbf{j}}|\sin\left(\frac{\pi}{2}\right)=1 \tag{A.3.9}$$

since the unit vector has magnitude $|\hat{\mathbf{i}}|=|\hat{\mathbf{j}}|=1$ and $\sin(\pi/2)=1$. By the right hand rule, the direction of $\hat{\mathbf{i}}\times\hat{\mathbf{j}}$ is in the $+\hat{\mathbf{k}}$ as shown in Figure A.3.4. Thus $\hat{\mathbf{i}}\times\hat{\mathbf{j}}=\hat{\mathbf{k}}$.

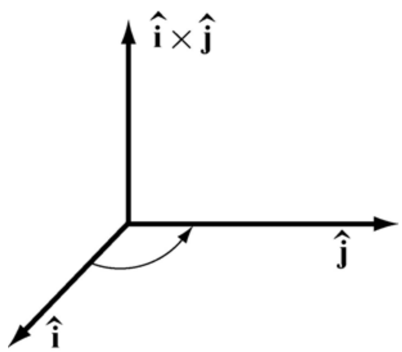

Figure A.3.4 Cross product of $\hat{\mathbf{i}}\times\hat{\mathbf{j}}$

We note that the same rule applies for the unit vectors in the *y* and *z* directions,

$$\hat{\mathbf{j}}\times\hat{\mathbf{k}}=\hat{\mathbf{i}},\quad \hat{\mathbf{k}}\times\hat{\mathbf{i}}=\hat{\mathbf{j}} \tag{A.3.10}$$

Note that by the anti-commutatively property (1) of the cross product,

$$\hat{\mathbf{j}}\times\hat{\mathbf{i}}=-\hat{\mathbf{k}},\quad \hat{\mathbf{i}}\times\hat{\mathbf{k}}=-\hat{\mathbf{j}} \tag{A.3.11}$$

The cross product of the unit vector $\hat{\mathbf{i}}$ with itself is zero because the two unit vectors are parallel to each other, ($\sin(0)=0$),

$$|\hat{\mathbf{i}} \times \hat{\mathbf{i}}| = |\hat{\mathbf{i}}||\hat{\mathbf{i}}|\sin(0) = 0 \tag{A.3.12}$$

The cross product of the unit vector $\hat{\mathbf{j}}$ with itself and the unit vector $\hat{\mathbf{k}}$ with itself, are also zero for the same reason.

$$\left|\hat{\mathbf{j}} \times \hat{\mathbf{j}}\right| = 0, \quad \left|\hat{\mathbf{k}} \times \hat{\mathbf{k}}\right| = 0 \tag{A.3.13}$$

With these properties in mind we can now develop an algebraic expression for the cross product in terms of components. Let's choose a Cartesian coordinate system with the vector $\vec{\mathbf{B}}$ pointing along the positive *x*-axis with positive *x*-component B_x. Then the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ can be written as

$$\vec{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}} \tag{A.3.14}$$

and

$$\vec{\mathbf{B}} = B_x\hat{\mathbf{i}} \tag{A.3.15}$$

respectively. The cross product in vector components is

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = (A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}) \times B_x\hat{\mathbf{i}} \tag{A.3.16}$$

This becomes, using properties (3) and (2),

$$\begin{aligned}\vec{\mathbf{A}} \times \vec{\mathbf{B}} &= (A_x\hat{\mathbf{i}} \times B_x\hat{\mathbf{i}}) + (A_y\hat{\mathbf{j}} \times B_x\hat{\mathbf{i}}) + (A_z\hat{\mathbf{k}} \times B_x\hat{\mathbf{i}}) \\ &= A_xB_x(\hat{\mathbf{i}} \times \hat{\mathbf{i}}) + A_yB_x(\hat{\mathbf{j}} \times \hat{\mathbf{i}}) + A_zB_x(\hat{\mathbf{k}} \times \hat{\mathbf{i}}) \\ &= -A_yB_x\hat{\mathbf{k}} + A_zB_x\hat{\mathbf{j}}\end{aligned} \tag{A.3.17}$$

The vector component expression for the cross product easily generalizes for arbitrary vectors

$$\vec{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}} \tag{A.3.18}$$

and

$$\vec{\mathbf{B}} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}} \tag{A.3.19}$$

to yield

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = (A_yB_z - A_zB_y)\hat{\mathbf{i}} + (A_zB_x - A_xB_z)\hat{\mathbf{j}} + (A_xB_y - A_yB_x)\hat{\mathbf{k}}\,. \tag{A.3.20}$$

The vector product

mc-TY-vectorprod-2009-1

One of the ways in which two vectors can be combined is known as the **vector product**. When we calculate the vector product of two vectors the result, as the name suggests, is a vector.

In this unit you will learn how to calculate the vector product and meet some geometrical applications.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- define the vector product of two vectors
- calculate the vector product when the two vectors are given in cartesian form
- use the vector product in some geometrical applications

Contents

1. Introduction

One of the ways in which two vectors can be combined is known as the **vector product**. When we calculate the vector product of two vectors the result, as the name suggests, is a vector.

In this unit you will learn how to calculate the vector product and meet some geometrical applications.

2. Definition of the vector product

Study the two vectors $\mathbf{a}$ and $\mathbf{b}$ drawn in Figure 1. Note that we have drawn the two vectors so that their tails are at the same point. The angle between the two vectors has been labelled θ.

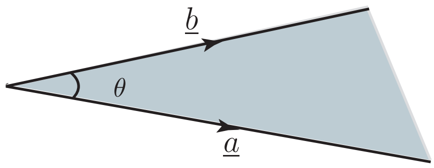

Figure 1. Two vectors $\mathbf{a}$ and $\mathbf{b}$ drawn so that the angle between them is θ.

As we stated before, when we find a **vector product** the result is a vector. We define the modulus, or magnitude, of this vector as

$$|\mathbf{a}|\,|\mathbf{b}|\,\sin\theta$$

so at this stage, a very similar definition to the scalar product, except now the sine of θ appears in the formula. However, this quantity is not a vector. To obtain a vector we need to specify a direction. By definition the direction of the vector product is such that it is at right angles to both $\mathbf{a}$ and $\mathbf{b}$. This means it is at right angles to the plane in which $\mathbf{a}$ and $\mathbf{b}$ lie. Figure 2 shows that we have two choices for such a direction.

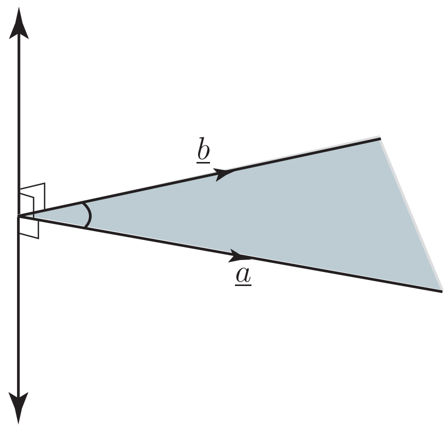

Figure 2. There are two directions which are perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

The convention is that we choose the direction specified by the right hand screw rule. This means that we imagine a screwdriver in the right hand. The direction of the vector product is the direction in which a screw would advance as the screwdriver handle is turned in the sense from $\mathbf{a}$ to $\mathbf{b}$. This is shown in Figure 3.

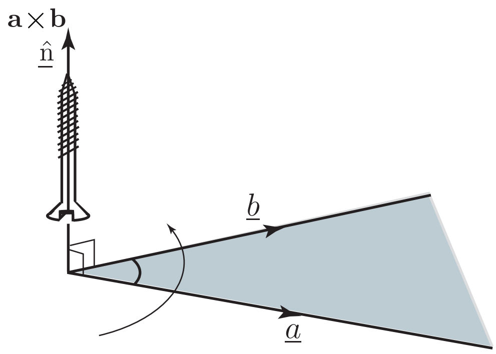

Figure 3. The direction of the vector product is determined by the right hand screw rule.

We let a unit vector in this direction be labelled $\hat{\mathbf{n}}$. We then define the vector product of $\mathbf{a}$ and $\mathbf{b}$ as follows:

Key Point

The **vector product** of $\mathbf{a}$ and $\mathbf{b}$ is defined to be

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,\sin\theta\,\hat{\mathbf{n}}$$

where

$|\mathbf{a}|$ is the modulus, or magnitude of $\mathbf{a}$,

$|\mathbf{b}|$ is the modulus of $\mathbf{b}$,

θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, and $\hat{\mathbf{n}}$ is a unit vector, perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ in a sense defined by the right hand screw rule.

Some people find it helpful to obtain the direction of the vector product using the right hand thumb rule. This is achieved by curling the fingers of the right hand in the direction in which $\mathbf{a}$ would be rotated to meet $\mathbf{b}$. The thumb then points in the direction of $\mathbf{a} \times \mathbf{b}$.

Yet another view is to align the first finger of the right hand with $\mathbf{a}$, and the middle finger with $\mathbf{b}$. If these two fingers and the thumb are then positiioned at right-angles, the thumb points in the direction of $\mathbf{a} \times \mathbf{b}$. Try this for yourself.

Note that the symbol for the vector product is the times sign, or cross $\times$, and so we sometimes refer to the vector product as the cross product. Either name will do. Some textbooks and some teachers and lecturers use the alternative 'wedge' symbol $\wedge$.

3. Some properties of the vector product

Suppose, for the two vectors $\mathbf{a}$ and $\mathbf{b}$ we calculate the product in a different order. That is, suppose we want to find $\mathbf{b} \times \mathbf{a}$. Using the definition of $\mathbf{b} \times \mathbf{a}$ and using the right-hand screw rule to obtain the required direction we find

$$\mathbf{b} \times \mathbf{a} = |\mathbf{b}|\,|\mathbf{a}|\,\sin\theta\,(-\hat{\mathbf{n}})$$

We see that the direction of $\mathbf{b} \times \mathbf{a}$ is opposite to that of $\mathbf{a} \times \mathbf{b}$ as shown in Figure 4. So

$$\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$$

So the vector product is **not commutative**. In practice, this means that the order in which we do the calculation does matter. $\mathbf{b} \times \mathbf{a}$ is in the opposite direction to $\mathbf{a} \times \mathbf{b}$.

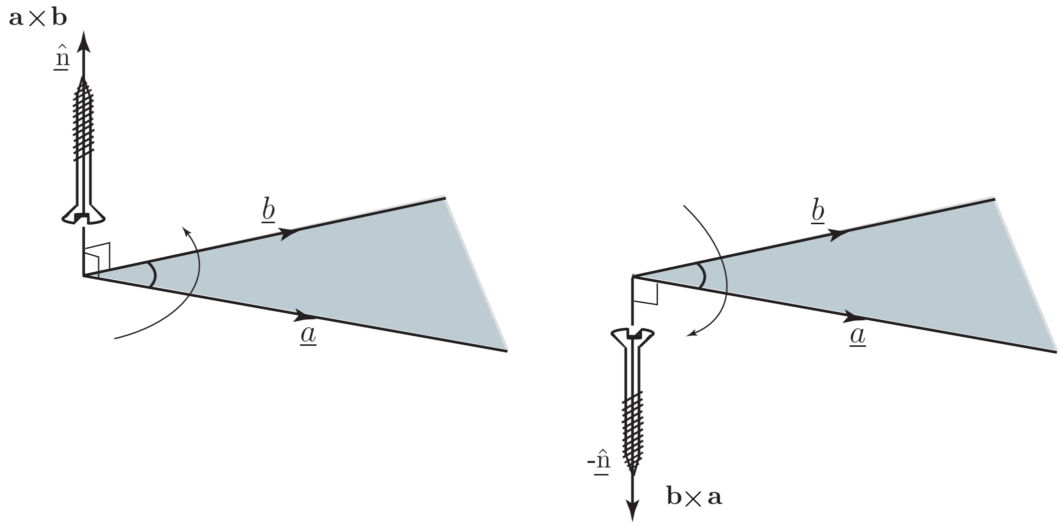

Figure 4. The direction of $\mathbf{b} \times \mathbf{a}$ is opposite to that of $\mathbf{a} \times \mathbf{b}$.

Key Point

The vector product is **not commutative**.

$$\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$$

Another property of the vector product is that it is **distributive over addition**. This means that

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

Although we shall not prove this result here we shall use it later on when we develop an alternative formula for finding the vector product.

Key Point

The vector product is **distributive over addition**. This means

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

Equivalently,

$$(\mathbf{b} + \mathbf{c}) \times \mathbf{a} = \mathbf{b} \times \mathbf{a} + \mathbf{c} \times \mathbf{a}$$

The vector product of two parallel vectors

Example

Suppose the two vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel. Strictly speaking the definition of the vector product does not apply, because two parallel vectors do not define a plane, and so it does not make sense to talk about a unit vector $\hat{n}$ perpendicular to the plane. But if we nevertheless write down the formula, we can see what the answer 'ought' to be:

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= |\mathbf{a}|\,|\mathbf{b}|\,\sin\theta\,\hat{\mathbf{n}} \\ &= |\mathbf{a}|\,|\mathbf{b}|\,\sin 0^\circ\,\hat{\mathbf{n}} \\ &= \mathbf{0} \end{aligned}$$

because $\sin 0^\circ = 0$. So, when two vectors are parallel we *define* their vector product to be the zero vector, $\mathbf{0}$.

Key Point

For two parallel vectors

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}$$

4. The vector product of two vectors given in cartesian form

We now consider how to find the vector product of two vectors when these vectors are given in cartesian form, for example as

$$\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + 7\mathbf{k} \qquad \text{and} \qquad \mathbf{b} = -5\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors in the directions of the x, y and z axes respectively.

First of all we need to develop a few results in the following examples.

Example

Suppose we want to find $\mathbf{i}\times\mathbf{j}$. The vectors $\mathbf{i}$ and $\mathbf{j}$ are shown in Figure 5. Note that because these vectors lie along the x and y axes they must be perpendicular.

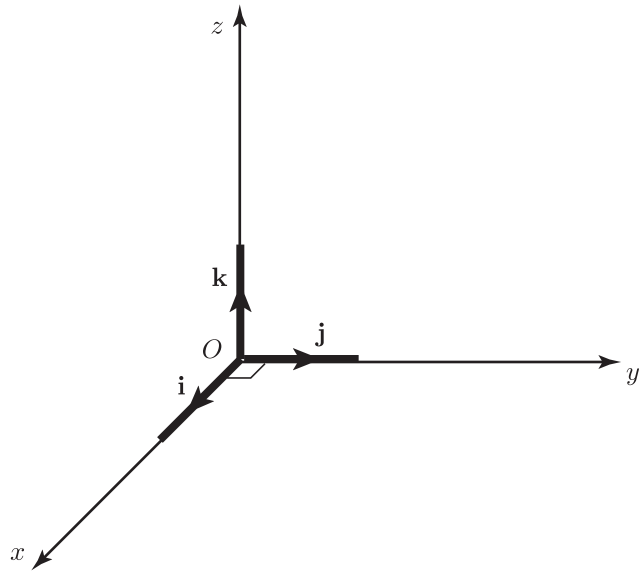

Figure 5. The unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. Note that $\mathbf{k}$ is a unit vector perpendicular to $\mathbf{i}$ and $\mathbf{j}$.

The angle between $\mathbf{i}$ and $\mathbf{j}$ is $90°$, and $\sin 90° = 1$. Further, if we apply the right hand screw rule, a vector perpendicular to both $\mathbf{i}$ and $\mathbf{j}$ is $\mathbf{k}$. Therefore

$$\begin{aligned}\mathbf{i}\times\mathbf{j} &= |\mathbf{i}|\,|\mathbf{j}|\,\sin 90°\,\mathbf{k}\\ &= (1)(1)(1)\,\mathbf{k}\\ &= \mathbf{k}\end{aligned}$$

Example

Suppose we want to find $\mathbf{j}\times\mathbf{i}$. Again, refer to Figure 5. If we apply the right hand screw rule, a vector perpendicular to both $\mathbf{j}$ and $\mathbf{i}$, in the sense defined by the right hand screw rule, is $-\mathbf{k}$. Therefore

$$\mathbf{j}\times\mathbf{i} = -\mathbf{k}$$

Example

Suppose we want to find $\mathbf{i}\times\mathbf{i}$. Because these two vectors are parallel the angle between them is $0°$. We can use the Key Point developed on page 5 to show that $\mathbf{i}\times\mathbf{i} = \mathbf{0}$.
In a similar manner we can derive all the results given in the following Key Point:

Key Point

$$\mathbf{i}\times\mathbf{i}=\mathbf{0} \qquad \mathbf{j}\times\mathbf{j}=\mathbf{0} \qquad \mathbf{k}\times\mathbf{k}=\mathbf{0}$$
$$\mathbf{i}\times\mathbf{j}=\mathbf{k} \qquad \mathbf{j}\times\mathbf{k}=\mathbf{i} \qquad \mathbf{k}\times\mathbf{i}=\mathbf{j}$$
$$\mathbf{j}\times\mathbf{i}=-\mathbf{k} \qquad \mathbf{k}\times\mathbf{j}=-\mathbf{i} \qquad \mathbf{i}\times\mathbf{k}=-\mathbf{j}$$

We can use these results to develop a formula for finding the vector product of two vectors given in cartesian form:

Suppose $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
&= a_1\mathbf{i} \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
&\qquad + a_2\mathbf{j} \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
&\qquad\qquad + a_3\mathbf{k} \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
&= a_1\mathbf{i} \times b_1\mathbf{i} + a_1\mathbf{i} \times b_2\mathbf{j} + a_1\mathbf{i} \times b_3\mathbf{k} \\
&\qquad + a_2\mathbf{j} \times b_1\mathbf{i} + a_2\mathbf{j} \times b_2\mathbf{j} + a_2\mathbf{j} \times b_3\mathbf{k} \\
&\qquad\qquad + a_3\mathbf{k} \times b_1\mathbf{i} + a_3\mathbf{k} \times b_2\mathbf{j} + a_3\mathbf{k} \times b_3\mathbf{k} \\
&= a_1b_1\mathbf{i} \times \mathbf{i} + a_1b_2\mathbf{i} \times \mathbf{j} + a_1b_3\mathbf{i} \times \mathbf{k} \\
&\qquad + a_2b_1\mathbf{j} \times \mathbf{i} + a_2b_2\mathbf{j} \times \mathbf{j} + a_2b_3\mathbf{j} \times \mathbf{k} \\
&\qquad\qquad + a_3b_1\mathbf{k} \times \mathbf{i} + a_3b_2\mathbf{k} \times \mathbf{j} + a_3b_3\mathbf{k} \times \mathbf{k}
\end{aligned}$$

Now, from the previous Key Point three of these terms are zero. Those that are not zero simplify to give

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

This is the formula which we can use to calculate a vector product when we are given the cartesian components of the two vectors.

Key Point

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

Example

Suppose we wish to find the vector product of the two vectors $\mathbf{a} = 4\mathbf{i}+3\mathbf{j}+7\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i}+5\mathbf{j}+4\mathbf{k}$.

We use the previous result with $a_1 = 4$, $a_2 = 3$, $a_3 = 7$ and $b_1 = 2$, $b_2 = 5$, $b_3 = 4$. Substitution into the formula gives

$$\mathbf{a} \times \mathbf{b} = ((3)(4) - (7)(5))\mathbf{i} + ((7)(2) - (4)(4))\mathbf{j} + ((4)(5) - (3)(2))\mathbf{k}$$

which simplifies to

$$\mathbf{a} \times \mathbf{b} = -23\mathbf{i} - 2\mathbf{j} + 14\mathbf{k}$$

For those familiar with evaluation of **determinants** there is a convenient way of remembering and representing this formula which is given in the following Key Point and which is explained in the accompanying video and in the Example below.

Key Point

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$$\begin{aligned}
\mathbf{a}\times\mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\
&= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k} \\
&= (a_2\times b_3 \;-\; a_3\times b_2)\mathbf{i} - (a_1\times b_3 \;-\; a_3\times b_1)\mathbf{j} + (a_1\times b_2 \;-\; a_2\times b_1)
\end{aligned}$$

Example

Suppose we wish to find the vector product of the two vectors $\mathbf{a} = 4\mathbf{i}+3\mathbf{j}+7\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i}+5\mathbf{j}+4\mathbf{k}$.

We write down a determinant, which is an array of numbers: in the first row we write the three unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. In the second and third rows we write the three components of $\mathbf{a}$ and $\mathbf{b}$ respectively:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 3 & 7 \\ 2 & 5 & 4 \end{vmatrix}$$

We then consider the first element in the first row, $\mathbf{i}$. Imagine covering up the elements in its row and column, to give the array $\begin{vmatrix} 3 & 7 \\ 5 & 4 \end{vmatrix}$. This is a so-called 2×2 determinant and is evaluated by finding the product of the elements on the leading diagonal (top left to bottom right) and subtracting the product of the elements on the other diagonal ($3\times 4 \;-\; 7\times 5 = -23$). The resulting number gives the $\mathbf{i}$ component of the final answer.

We then consider the second element in the first row, $\mathbf{j}$. Imagine covering up the elements in its row and column, to give the array $\begin{vmatrix} 4 & 7 \\ 2 & 4 \end{vmatrix}$. This 2×2 determinant is evaluated, as before, by finding the product of the elements on the leading diagonal (top left to bottom right) and subtracting the product of the elements on the other diagonal, ($4\times 4 \;-\; 7\times 2 = 2$). The result is then multiplied by -1 and this gives the $\mathbf{j}$ component of the final answer, that is -2.

Finally, we consider the third element in the first row, $\mathbf{k}$. Imagine covering up the elements in its row and column, to give the array $\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}$. This determinant is evaluated, as before, by finding

the product of the elements on the leading diagonal (top left to bottom right) and subtracting the product of the elements on the other diagonal ($4 \times 5 \; - \; 3 \times 2 = 14$). The resulting number gives the $\mathbf{k}$ component of the final answer.
We write all this as follows:

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 3 & 7 \\ 2 & 5 & 4 \end{vmatrix} \\
&= \begin{vmatrix} 3 & 7 \\ 5 & 4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 4 & 7 \\ 2 & 4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix} \mathbf{k} \\
&= (3 \times 4 \; - \; 7 \times 5)\mathbf{i} - (4 \times 4 \; - \; 7 \times 2)\mathbf{j} + (4 \times 5 \; - \; 3 \times 2)\mathbf{k} \\
&= -23\mathbf{i} - 2\mathbf{j} + 14\mathbf{k}
\end{aligned}$$

Exercises 1

1. Use the formula $\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$ to find the vector product $\mathbf{a} \times \mathbf{b}$ in each of the following cases.

(a) $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = -2\mathbf{i} + 9\mathbf{j}$.
(b) $\mathbf{a} = 4\mathbf{i} - 2\mathbf{j}$, $\mathbf{b} = 5\mathbf{i} - 7\mathbf{j}$.

Comment upon your solutions.
2. Use the formula in Q1 to find the vector product $\mathbf{a} \times \mathbf{b}$ in each of the following cases.

(a) $\mathbf{a} = 5\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 8\mathbf{j} + 9\mathbf{k}$.
(b) $\mathbf{a} = \mathbf{i} + \mathbf{j} - 12\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$.

3. Use determinants to find the vector product $\mathbf{p} \times \mathbf{q}$ in each of the following cases.

(a) $\mathbf{p} = \mathbf{i} + 4\mathbf{j} + 9\mathbf{k}$, $\mathbf{q} = 2\mathbf{i} - \mathbf{k}$.
(b) $\mathbf{p} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{q} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$.

4. For the vectors $\mathbf{p} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{q} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$ show that, in this special case, $\mathbf{p} \times \mathbf{q} = \mathbf{q} \times \mathbf{p}$.
5. For the vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, $\mathbf{c} = 7\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, show that

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$$

5. Some applications of the vector product

In this section we will look at some ways in which the vector product can be used.

Using the vector product to find a vector perpendicular to two given vectors.

One of the common applications of the vector product is to finding a vector which is perpendicular to two given vectors. The two vectors should be non-zero and must not be parallel.

Example

Suppose we wish to find a vector which is perpendicular to both of the vectors $\mathbf{a} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 5\mathbf{i} - 3\mathbf{k}$.

We know from the definition of the vector product that the vector $\mathbf{a} \times \mathbf{b}$ will be perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. So first of all we calculate $\mathbf{a} \times \mathbf{b}$.

$$\begin{aligned}
\mathbf{a}\times\mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -2 \\ 5 & 0 & -3 \end{vmatrix} \\
&= (3\times -3\ -\ (-2)\times 0)\mathbf{i} - (1\times -3\ -\ (-2)\times 5)\mathbf{j} + (1\times 0\ -\ 3\times 5)\mathbf{k} \\
&= -9\mathbf{i} - 7\mathbf{j} - 15\mathbf{k}
\end{aligned}$$

This vector is perpendicular to $\mathbf{a}$ and $\mathbf{b}$.
On occasions you may be asked to find a unit vector which is perpendicular to two given vectors. To convert a vector into a unit vector in the same direction we must divide it by its modulus. The modulus of $-9\mathbf{i} - 7\mathbf{j} - 15\mathbf{k}$ is

$$|\mathbf{a}\times\mathbf{b}| = \sqrt{(-9)^2 + (-7)^2 + (-15)^2} = \sqrt{355}$$

So, finally, the required unit vector is $\dfrac{1}{\sqrt{355}}(-9\mathbf{i} - 7\mathbf{j} - 15\mathbf{k})$.

Using the vector product to find the area of a parallelogram.

Consider the parallelogram shown in Figure 6 which has sides given by vectors $\mathbf{b}$ and $\mathbf{c}$.

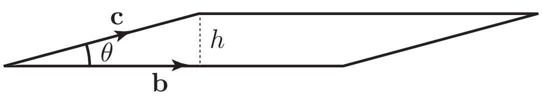

Figure 6. A parallelogram with two sides given by $\mathbf{b}$ and $\mathbf{c}$.

The area of the parallelogram is the length of the base multiplied by the perpendicular height, h. Now $\sin\theta = \dfrac{h}{|\mathbf{c}|}$ and so $h = |\mathbf{c}|\sin\theta$. Therefore

$$\text{area} = |\mathbf{b}|\,|\mathbf{c}|\sin\theta$$

which is simply the modulus of the vector product of $\mathbf{b}$ and $\mathbf{c}$. We deduce that the area of the parallelogram is given by

$$\text{area} = |\mathbf{b}\times\mathbf{c}|$$

Using the vector product to find the volume of a parallelepiped.

Consider Figure 7 which illustrates a parallelepiped. This is a six sided solid, the sides of which are parallelograms. Opposite parallelograms are identical. The volume, V, of a parallelepiped with edges $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is given by

$$V = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$$

This formula can be obtained by understanding that the volume is the product of the area of the base and the perpendicular height. Because the base is a parallelogram its area is $|\mathbf{b}\times\mathbf{c}|$. The perpendicular height is the component of $\mathbf{a}$ in the direction perpendicular to the plane containing $\mathbf{b}$ and $\mathbf{a}$, and this is $h = \mathbf{a}\cdot\widehat{\mathbf{b}\times\mathbf{c}}$. So the volume is given by

$$\begin{aligned}
V &= \text{(height)(area of base)} \\
&= \mathbf{a}\cdot\widehat{\mathbf{b}\times\mathbf{c}}\ |\mathbf{b}\times\mathbf{c}| \\
&= \mathbf{a}\cdot\frac{\mathbf{b}\times\mathbf{c}}{|\mathbf{b}\times\mathbf{c}|}\ |\mathbf{b}\times\mathbf{c}| \\
&= \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})
\end{aligned}$$

This could turn out to be negative, so in fact, for the volume we take its modulus: $V = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$.

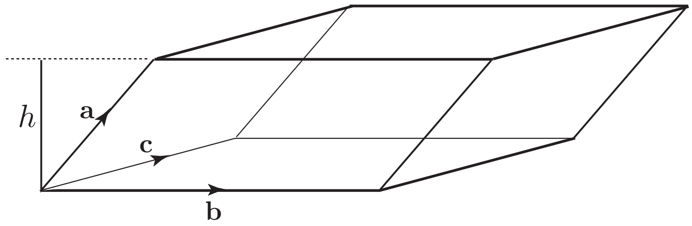

Figure 7. A parallelepiped with edges given by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Example

Suppose we wish to find the volume of the parallelepiped with edges $\mathbf{a} = 3\mathbf{i}+2\mathbf{j}+\mathbf{k}$, $\mathbf{b} = 2\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\mathbf{c} = \mathbf{i}+2\mathbf{j}+4\mathbf{k}$.

We first evaluate the vector product $\mathbf{b}\times\mathbf{c}$.

$$\begin{aligned}
\mathbf{b}\times\mathbf{c} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ 1 & 2 & 4 \end{vmatrix} \\
&= (1\times 4 \;-\; 1\times 2)\mathbf{i} - (2\times 4 \;-\; 1\times 1)\mathbf{j} + (2\times 2 \;-\; 1\times 1)\mathbf{k} \\
&= 2\mathbf{i} - 7\mathbf{j} + 3\mathbf{k}
\end{aligned}$$

Then we need to find the scalar product of $\mathbf{a}$ with $\mathbf{b}\times\mathbf{c}$.

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (3\mathbf{i}+2\mathbf{j}+\mathbf{k})\cdot(2\mathbf{i}-7\mathbf{j}+3\mathbf{k}) = 6 - 14 + 3 = -5$$

Finally, we want the modulus, or absolute value, of this result. We conclude the parallelepiped has volume 5 (units cubed).

Exercises 2.

1. Find a unit vector which is perpendicular to both $\mathbf{a} = \mathbf{i}+2\mathbf{j}-3\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i}+3\mathbf{j}+\mathbf{k}$.

2. Find the area of the parallelogram with edges represented by the vectors $2\mathbf{i}-\mathbf{j}+3\mathbf{k}$ and $7\mathbf{i}+\mathbf{j}+\mathbf{k}$.

3. Find the volume of the parallelepiped with edges represented by the vectors $\mathbf{i}+\mathbf{j}+\mathbf{k}$, $2\mathbf{i}+3\mathbf{j}+4\mathbf{k}$ and $3\mathbf{i}-2\mathbf{j}+\mathbf{k}$.

4. Calculate the **triple scalar product** $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ when $\mathbf{a} = 2\mathbf{i}-2\mathbf{j}+\mathbf{k}$, $\mathbf{b} = 2\mathbf{i}+\mathbf{j}$ and $\mathbf{c} = 3\mathbf{i}+2\mathbf{j}+\mathbf{k}$.

Answers to Exercises

Exercises 1.

1. (a) $24\mathbf{k}$, (b) $-18\mathbf{k}$. Both answers are vectors in the z direction. The given vectors, $\mathbf{a}$ and $\mathbf{b}$, lie in the xy plane.
2. (a) $59\mathbf{i}-37\mathbf{j}-46\mathbf{k}$, (b) $13\mathbf{i}-25\mathbf{j}-\mathbf{k}$.
3. (a) $-4\mathbf{i}+19\mathbf{j}-8\mathbf{k}$, (b) $-\mathbf{i}+10\mathbf{j}-7\mathbf{k}$.
4. Both cross products equal zero, and so, in this special case $\mathbf{p}\times\mathbf{q} = \mathbf{q}\times\mathbf{p}$. The two given vectors are anti-parallel.
5. Both equal $-11\mathbf{i}+25\mathbf{j}-13\mathbf{k}$.

Exercises 2.

1. $\frac{1}{\sqrt{171}}(11\mathbf{i} - 7\mathbf{j} - \mathbf{k})$.
2. $\sqrt{458}$ square units.
3. 8 units cubed.
4. 7.

